New Method for Propagating the Square Root Covariance Matrix in Triangular Form

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Introduction

ONE of the problems encountered in applying the Kalman-Bucy[1] filter to the problem of estimating the state of a dynamical system, lies in the loss of numerical significance in the calculation of the state error covariance matrix. The loss of significance, usually associated with processing a number of highly accurate observations during a short time span, often destroys the positive definite character of the covariance matrix, and filter instability usually results. A method proposed by Potter[2] is described, in which the square-root of the state error covariance matrix is used to process the observations. The method, which is restricted to process only scalar observations, has the advantage that the covariance matrix is always nonnegative definite, and since the computational procedure uses the square root of the covariance matrix, it requires only one-half of the significant digits required to compute the state error covariance matrix. Bellantoni and Dodge[3] extended the method to allow the processing of a p-vector of simultaneous measurements. However, this approach requires a p × p matrix diagonalization at each observation epoch and, consequently, is not competitive with the conventional Kalman-Bucy filter from the point of view of computing time or program complexity. Kaminsky et al.,[4] gives an excellent summary of the square root filtering methods.

An improvement to Potter's original square root filter algorithm has been proposed recently by Carlson.[5] Carlson's method utilizes an analytic Cholesky decomposition algorithm to maintain the covariance square root matrix in triangular form during the incorporation of observations. The triangular form of the square root matrix provides significant computational economy during the subsequent covariance update. However, the method proposed by Carlson requires the use of a lower triangular form for the a priori or propagated state error covariance square root $\bar{W}$.

There are three basic methods for achieving this objective. The first, and generally the faster of the methods, employs the Cholesky decomposition. The covariance square root matrix is propagated between observation intervals using the state transition matrix. However, the propagated square root will not be triangular, even if the initial matrix is triangular. To obtain a triangular square root, the covariance matrix is reformed and the triangular square root is obtained after the process noise is added by using the Cholesky decomposition. The second method is known as the Housholder triangularization procedure. This procedure yields the desired lower triangular form for the propagated square root matrix $\bar{W}$ directly. This method usually yields more accurate results than Cholesky decomposition procedures; however, it is considerably slower. The third method for propagating the covariance square root in triangular form is known as the Gram-Schmidt orthogonalization procedure. This method requires about the same computation time as the Housholder triangularization procedure. These methods are discussed in detail in Ref. 4.

Each of the three techniques discussed uses a discrete propagation which involves the computation of the state transition matrix. A method for propagating the square root of the covariance matrix based on the matrix Riccati equation which governs the state-error covariance matrix has been proposed by Andrews.[6] This technique has the disadvantage that the inverse of the covariance square root must be computed at each integration step. Consequently, use of either a discrete or a continuous algorithm for covariance matrix propagation fails to achieve the maximum advantage of the triangular square root measurement update algorithm.

In this investigation, a new method which propagates the covariance square root matrix in lower triangular form is given for the discrete observation case. The method is faster than the previously proposed algorithms and is well adapted for use with the Carlson square-root measurement algorithm.

Deriviation of the Algorithm

In the Kalman-Bucy filter, the equation for updating the estimation error covariance matrix P_k at each observation epoch can be expressed as follows

$$P_k = \bar{P}_k - \bar{P}_k H_k^T (H_k \bar{P}_k H_k^T + R_k)^{-1} H_k \bar{P}_k \tag{1}$$

Between observations, the matrix propagates according to the differential equation

$$\dot{\bar{P}} = A\bar{P} + \bar{P}A^T + Q \tag{2}$$

where Q is the process noise covariance matrix and, in this discussion, it is assumed to be a diagonal matrix. (This assumption is not a necessary restriction for the derivation.)

Experience has shown that the numerical solution of Eqs. (1) and (2) often results in a negative definite covariance matrix followed by the occurrence of numerical stability problems. This condition can be avoided by replacing the covariance matrix P_k and $\bar{P}_k$ with their square roots. For compatibility, it is also desirable that the covariance matrix be propagated in square-root form. To achieve this objective, we define the matrix W such that

$$\bar{P} = WW^T \tag{3}$$

By taking the time derivative of Eq. (3), the following relationship is obtained

$$\dot{\bar{P}} = \dot{W}W^T + W\dot{W}^T \tag{4}$$

Substitution of Eqs. (3) and (4) into Eq. (2) yields the following expression

$$\dot{W}W^T + W\dot{W}^T = AWW^T + WW^TA^T + Q^* + Q^{*T} \tag{5}$$

where $Q^* = Q/2$. By regrouping terms, Eq. (5) can be expressed as

$$[\dot{W}W^T - AWW^T - Q^*] + [W\dot{W}^T - WW^TA^T - Q^{*T}] = 0 \tag{6}$$

It can be easily seen that the matrix inside the first bracket of Eq. (6) is the transpose of the expression in the second bracket. Therefore the general solution of the matrix Eq. (6) is given by

$$\dot{W}W^T - AWW^T - Q^* = C^* \tag{7}$$

where C^* is an arbitrary skew symmetric matrix such that for $i, j = 1 \ldots n$,

$$C_{ii}^* = 0, \quad C_{ij}^* = -C_{ji}^* \quad \text{for} \quad i \neq j \tag{8}$$

If C^* is skew symmetric, then the following relationship is always true

$$C^* + C^{*T} = 0 \tag{9}$$

Note that Eq. (7), with $C^* = 0$, is a particular solution to the matrix equation given in Eq. (6). Assuming that the inverse of W^T exists, Eq. (7) becomes

$$\dot{W} = AW + (Q^* + C^*)W^{-T} \tag{10}$$

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where $W^{-T} = (W^T)^{-1}$. Over a small time interval Δt the solution to Eq. (10) can be approximated by the following expression:

$$W(t+\Delta t) = W(t) + \Delta t \dot{W}(t) \tag{11}$$

Substitution of Eq. (10) for $\dot{W}(t)$ in Eq. (11) leads to the following result

$$W(t+\Delta t) = (I + \Delta t A) W(t) + \Delta t (Q^* + C^*) W(t)^{-T} \tag{12}$$

With the following definitions

$$W_{k+1} = W(t+\Delta t); \qquad W_k = W(t)$$
$$Q_k = \Delta t Q^*; \qquad C_k = \Delta t C^* \tag{13}$$

Eq. (12) can be approximated for small Δt, as follows

$$W_{k+1} = \Phi(t_{k+1}, t_k) W_k + (Q_k + C_k) W_k^{-T} \tag{14}$$

where $\Phi(t_{k+1}, t_k)$ is the state transition matrix, and the matrix Q_k is one-half of the discrete process noise covariance matrix. Also note that the matrix C_k is still skew symmetric. Equation (14) can be used to propagate the W_k matrix discretely in time. The usual solutions to Eq. (14) do not maintain the $\bar{W}$ matrix in triangular form. However, it is possible to find the particular skew symmetric matrix C_k such that $\bar{W}_{k+1}$ will remain in lower triangular form. The real disadvantage of applying this aproach to Eq. (14) is that it requires the computation of the inverse of W_k^T. Even though the square root matrix W_k is lower triangular, this is a costly operation in terms of computer execution time. Using an alternate aproach, the particular matrix C_k which will maintain $\bar{W}_k$ in lower triangular form can be developed as follows.

Omitting the subscripts k, Eq. (14) can be rewritten as follows

$$[\bar{W} - \Phi W] W^T = Q + C \tag{15}$$

Let $T = \Phi W$ and $V = \bar{W} - T$. Then, if W is lower triangular and if $\bar{W}$ is required to be lower triangular, the elements of V will satisfy the following relations

$$V_{ij} = \begin{matrix} -T_{ij}, & \text{for } j > i \\ \bar{W}_{ij} - T_{ij}, & \text{for } i \geq j \end{matrix} \tag{16}$$

Therefore Eq. (15) becomes

$$VW^T = Q + C \tag{17}$$

Since C and $\bar{W}$ are skew symmetric and lower triangular matrices respectively, Eq. (17) contains n^2 unknowns, i.e., the C_{ij} and $\bar{W}_{ij}$, and exactly n^2 equations to determine the unknowns. Note, however, that Eq. (16) is of special form. That is, the unknowns are found on both sides of the equality sign.

In terms of matrix elements, Eq. (17) can be expressed as follows

$$\begin{bmatrix} V_{11} & -T_{12} & -T_{13} \cdots & -T_{1n} \\ V_{21} & V_{22} & -T_{23} \cdots & \vdots \\ V_{31} & V_{32} & V_{33} \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & \cdots & \cdots & V_{nn} \end{bmatrix} \begin{bmatrix} W_{11} & W_{21} & W_{31} \cdots & W_{n1} \\ 0 & W_{22} & W_{32} \cdots & \vdots \\ \vdots & \vdots & W_{33} \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & W_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} Q_1 & C_{12} & C_{13} \cdots & C_{1n} \\ -C_{12} & Q_2 & C_{23} \cdots & \vdots \\ -C_{13} & -C_{23} & Q_3 \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -C_{1n} & \cdots & \cdots & Q_n \end{bmatrix} \tag{18}$$

The solution of Eq. (18) can be obtained sequentially starting from the (1, 1) component, i.e., first solve for V_{11} and then determine C_{1j}, for $j = 2, n$. Then the C_{1j}, $j = 2 \ldots, n$ can be used to determine V_{i1}, $i = 2, n$. The element V_{22} is found using the (2, 2) components on both sides of the equality sign. Once V_{22} is found, C_{2j}, $j = 3$, n can be found and so can the elements V_{i2}, $i = 3$, n. This process continues until all of the unknown components of V are found. The solution obtained by this process can be summarized as follows

1st column of $\bar{W}$ and 1st row of C

$$V_{11} = Q_1 / W_{11}$$
$$\bar{W}_{11} = V_{11} + T_{11}$$
$$C_{12} = V_{11} W_{21} - T_{12} W_{22}$$
$$V_{21} = -C_{12} / W_{11}$$
$$\bar{W}_{21} = V_{21} + T_{21}$$
$$\vdots$$
$$C_{1n} = \sum_{k=1}^{n} (\tilde{V}_{1k} \cdot W_{nk})$$
$$V_{n1} = -C_{1n} / W_{11}$$
$$\bar{W}_{n1} = V_{n1} + T_{n1}$$

2nd column of $\bar{W}$ and 2nd row of C

$$V_{22} = [Q_2 - V_{21} W_{21}] / W_{22}$$
$$\bar{W}_{22} = V_{22} + T_{22}$$
$$C_{23} = V_{21} W_{31} + V_{22} W_{33} - T_{23} W_{33}$$
$$V_{32} = -[C_{23} + V_{31} W_{21}] / W_{22}$$
$$\bar{W}_{32} = V_{32} + T_{32}$$
$$\vdots$$
$$C_{2n} = \sum_{k=1}^{n} V_{2k} \cdot W_{nk}$$
$$V_{n2} = -[C_{2n} / W_{22} + V_{n1} W_{n1} / W_{22}]$$
$$\bar{W}_{n2} = V_{n2} + T_{n2}$$

By this procedure, *the jth column of* $\bar{W}$ can be expressed as follows

$$V_{jj} = \left(Q_j - \sum_{k=1}^{j-1} V_{jk} W_{jk} \right) \Big/ W_{jj}, \qquad j = 2, n \tag{19}$$

$$\bar{W}_{jj} = V_{jj} + T_{jj} \tag{20}$$

$$V_{ij} = -\left(\sum_{k=1}^{i} V_{jk} W_{ik} + \sum_{k=1}^{j-1} V_{ik} W_{jk} \right) \Big/ W_{jj}, \qquad i > j \tag{21}$$

$$\bar{W}_{ij} V_{ij} + T_{ij} \tag{22}$$

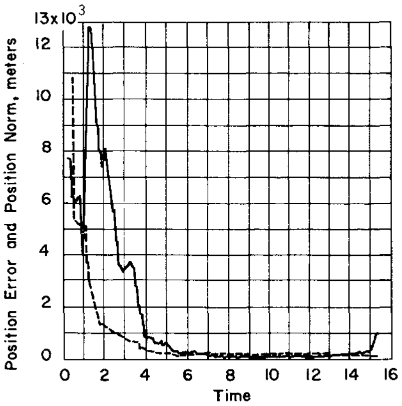
Equations (19–22) can be applied sequentially to find the elements of the propagated covariance square root matrix $\bar{W}$.

Examination of Eqs. (19–22) indicates that there are three basic advantages of this method for propagating the W matrix: first, the inverse of W is not required; second, the covariance matrix P is never reformed to compute $\bar{W}$; and third, the algorithm requires no extra storage for the C matrix. As a consequence of these factors, the algorithm is faster than any other reported algorithm for propagating the square root of the covariance matrix in lower triangular form.

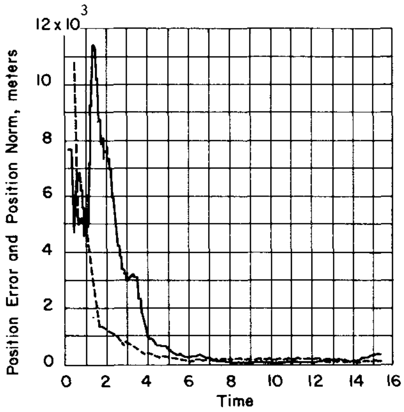
It should be noted that the algorithm derived from Eq. (12) is restricted to discrete observation applications where the time interval between observations, Δt, is small enough to allow the approximations used in Eq. (11) to be valid. The approximations leading to Eq. (12) are not essential, however. The approach used in determining Eqs. (19–22) from Eq. (12) can be applied using Eq. (10) as the starting point to obtain a set of differential equations for the square-root covariance matrix which are lower-triangular in form. The equations which result from this approach are discussed in Ref. 7. The set of equations given in this report are applicable for applications in which the state transition matrix is to be integrated. One such application which is considered in the next section, is the re-entry navigation filter for the space-shuttle.

Numerical Comparison

The performance of the algorithm described in the previous section was evaluated during a simulated study of the problem of estimating the state of the space-shuttle during the period from black-out exit at an altitude of 140,000 ft to touch down. The simulated observations used in the study were one-way doppler observations sampled at $\frac{1}{2}$ sec intervals. The standard deviation assumed for the observations was $\sigma_p = 0.04$ m. The equations of



a) Covariance Algorithm



b) Covariance Square-Root Algorithm

Fig. 1 Comparison of square-root estimates with covariance (Kalman-Bucy) estimates.

motion assumed for the comparison are a realistic approximation to the equations which govern the shuttle vehicle during the post-blackout maneuver. A complete description of the equations of motion and the observation-state relations used in the simulation described here is given in Ref. 8. In this investigation, the triangular state-error covariance algorithm discussed in the previous section was combined with a Carlson square-root measurement update algorithm[5] to obtain a complete filter algorithm. This algorithm was implemented in the computer simulation program described in Ref. 8. Figure 1 shows the variation with time of the position error norm $\Delta r = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$ and the associated state-error covariance matrix norm $(P_{xx} + P_{yy} + P_{zz})^{1/2}$ as determined by the conventional Kalman-Bucy filter. In Fig. 1b, the error norm and the covariance matrix norm obtained with the square-root filter are shown. The covariance matrix P for the square-root filter was obtained using the relation $\bar{P} = WW^T$, i.e., Eq. (3). Note that the norm of the covariance matrix position elements is essentially identical for both methods. There are slight differences in the position error norm estimates prior to the 3 min epoch. After this time period, the position error estimates for both algorithms are essentially the same. Similiar results were obtained for the estimates of the velocity components. Finally the investigation indicates that, using the UNIVAC 1107 at the NASA Johnson Spacecraft Center, the square-root filter algorithm obtained by combining the triangular covariance matrix progation algorithm with the triangular measurement update algorithm described in Ref. 5 requires a computation time which exceeds that required by the standard Kalman-Bucy filter by less than 15%.

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Velocity and Shear-Stress in a Transpired Turbulent Boundary Layer

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Introduction

In recent years there has been considerable effort devoted to the study of turbulent boundary-layer flows over porous surfaces with mass transfer. There have been methods developed for calculating such flows, but these studies have had to rely upon extensive numerical schemes, or only yielded information about the gross features of the boundary layer such as momentum or displacement thickness. The present method presents a simple means of predicting not only the gross parameters, but also distributions of velocity and Reynolds stress through the turbulent boundary layer.

Theory and Results

For a two-dimensional turbulent boundary-layer flow with zero pressure gradient, the governing equations of continuity and momentum are given by

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \tag{1}$$

$$u(\partial u/\partial x) + v(\partial u/\partial y) = (1/\rho)(\partial \tau/\partial y) \tag{2}$$

where ρ is the fluid density, and τ is the sum of the laminar shear stress τ_l and the turbulent shear stress τ_t.

The boundary conditions to be imposed upon Eqs. (1) and (2) are given by

$$u(x,0) = 0, \qquad u(x,\delta) = U_\infty, \qquad v(x,0) = V_w \tag{3}$$

where δ and U_∞ are the boundary-layer thickness and freestream velocity, respectively. V_w is the transpiration velocity at the wall. It is assumed that a similarity solution to Eqs. (1) and (2) exists, and has the form

$$u(x, y) = U_\infty f(\eta) \qquad \eta(x, y) = y/\delta(x) \tag{4}$$

Using Eq. (4) and the momentum integral equation, streamwise derivatives may be written as

$$\partial/\partial x = -(C_f/2\theta)(1+B)\eta(d/d\eta) \tag{5}$$

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